Jacobians of Compact Riemann Surfaces

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Motivation: Why should we consider compact Riemann surfaces? Let A be an abelian variety over $\mathbb{C}$. Then $A(\mathbb{C})$ inherits a complex structure as submanifold of $\mathbb{P}^n(\mathbb{C})$. It is a connected compact complex manifold and has an abelian group structure. Note that $A(\mathbb{C}) \cong V/\Lambda$ where $V \cong \mathbb{C}^g$ and $\Lambda \cong \mathbb{Z}^{2g}$.

Let C be a smooth projective connected curve over $\mathbb{C}$. Then $C(\mathbb{C})$ is a compact connected Riemann surface. So we have

$$\begin{tikzcd}
 & C \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & C(\mathbb{C}) \arrow[drr, rightsquigarrow] & & \\
 & J_C \arrow[r, rightsquigarrow] & J_C(\mathbb{C}) \cong V/\Lambda & \cong & J(C(\mathbb{C})) \\
C \arrow[ur, "AJ"] & & & C(\mathbb{C}) \arrow[ur, "AJ"] &
\end{tikzcd}$$

where $J(X)$ for X a Riemann surface is to be defined. We want to complete the analytic side.

Remark: Every compact connected Riemann surface is an algebraic curve. [2]

For the duration of the talk, unless stated otherwise, X is a compact connected Riemann surface of genus g.

Goals: (i) Define the Jacobian of X, $J(X)$.
(ii) Describe its structure as a complex torus. We will view it as the quotient $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}_X)$ which is isomorphic to $\mathbb{C}^g/\Lambda$ for some lattice Λ.
(iii) Classify line bundles on X.
(iv) Define the first Chern class and use it to identify the Jacobian with a torus.
(v) Show that $J(X) \cong H^0(X,\Omega^1_X)^*/H_1(X,\mathbb{Z})$ if time permits.
(vi) Play with the Abel-Jacobi map if we have the time.

Definitions: (i) $\mathrm{Div}(X)$ is the free abelian group of points in X.

(ii) A divisor is called principal if it equals

$$\mathrm{div}(f) = \sum_p \mathrm{ord}_p(f) \cdot p$$

where $\mathbb{C}(X)$ is meromorphic functions on X, $\mathrm{ord}_p(f)$ is the order of the pole/zero of $f(p)$, and $f \in \mathbb{C}(X)^\times$. Notice that div is a homomorphism

$$\mathbb{C}(X)^\times \longrightarrow \mathrm{Div}(X)$$

and hence $\mathrm{Princ}(X)$ is a subgroup of $\mathrm{Div}(X)$.

(iii) We call

$$\mathrm{C}\ell(X) = \mathrm{Div}(X) / \mathrm{Princ}(X)$$

the divisor class group.

(iv) $\mathrm{Div}^0(X)$ is the kernel of the degree map, i.e. $\deg : \mathrm{Div}(X) \to \mathbb{Z}$ $\quad \Sigma_k n_k p_k \longmapsto \Sigma_k n_k$. It is a non-trivial result that principal divisors have degree zero. So we have that $\mathrm{Princ}(X)$ is a subgroup of $\mathrm{Div}^0(X)$ as well.

(v) Denote

$$\mathrm{C}\ell^0(X) = \mathrm{Div}^0(X) / \mathrm{Princ}(X).$$

We then have

$$0 \longrightarrow \mathrm{C}\ell^0(X) \longrightarrow \mathrm{C}\ell(X) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Theorem: $Cl^0(X)$ can be given the structure of a g-dimensional complex torus, i.e. the quotient of a g-dimensional complex vector space by a lattice.

Definition: The Jacobian of X, $J(X)$, is $Cl^0(X)$ together with the above structure.

Our strategy for the proof is:
(i) Show $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}_X)$ is a torus in $H^1(X,\mathcal{O}_X^*)$.
(ii) Show this torus is exactly $Cl^0(X)\subset H^1(X,\mathcal{O}_X^*)$. This will be done by means of showing that both identify with the kernel of the Chern class map

$$c_1: H^1(X,\mathcal{O}_X^*)\longrightarrow H^2(X,\mathbb{Z})\cong\mathbb{Z}.$$

Definitions: (i) A (differential) form on X is a section of the exterior algebra of the cotangent bundle over X. Differential 2-forms can be integrated over X.

(ii) A form α is exact if there exists a form β such that $\alpha=d\beta$

(iii) A form α is closed if $d\alpha=0$.

(iv) The de Rham complex is the cochain complex

$$0\to\Omega^0(X)\xrightarrow{d}\Omega^1(X)\xrightarrow{d}\Omega^2(X)\longrightarrow\cdots$$

where $\Omega^i(X)$ is smooth functions on X for $i=0$ and i-forms on X for $i>0$. The differential is defined by (a) $d\alpha$ is the differential of α for a 0-form α, (b) $dd\alpha=0$ for α a 0-form, (c) $d(\alpha\wedge\beta)=d\alpha\wedge\beta+$ $+(-1)^i(\alpha\wedge d\beta)$.

The cohomology of this complex is the de Rham cohomology. $H^i_{dR}\cong\{\text{closed } i\text{-forms}\}/\{\text{exact } i\text{-forms}\}$.

As a first step we must classify line bundles on X. Let $\mathcal{M}$ be a line bundle on X with $\phi_i:\mathcal{M}(\mathcal{U}_i)\xrightarrow{\cong}\mathcal{O}_X(\mathcal{U}_i)$ local trivialization. Thus $\phi_i\phi_j^{-1}:\mathcal{O}_X(\mathcal{U}_{ij})\xrightarrow{\cong}\mathcal{O}_X(\mathcal{U}_{ij})$. Hence $\phi_i\phi_j^{-1}$ is given by $f_{ij}\in\mathcal{O}_X(\mathcal{U}_{ij})^\times$, i.e. f_{ij} is a nowhere zero holomorphic function. We see that

$$f_{ij}f_{jk}f_{ki}=\phi_i\phi_j^{-1}\phi_j\phi_k^{-1}\phi_k\phi_i^{-1}=1.$$

Therefore f_{ij} defines a 1-cocycle for $\mathcal{O}_X^\times$.

We have the following isomorphisms

$$\{f_{ij}\}/\sim\;=\check{H}^1(X,\mathcal{O}_X^\times)\xleftarrow{\cong}H^1(X,\mathcal{O}_X^\times)\cong\{\mathcal{O}_X^\times\text{-torsors}\}.$$

Let P be an $\mathcal{O}_X^\times$-torsor. So we have the action of $\mathcal{O}_X^\times$ on P. Then we have a line bundle

$$P\times^{\mathcal{O}_X^\times}\mathcal{O}_X\in \mathrm{Pic}(X).$$

Next assume we have a line bundle, $\mathcal{M}$. We map $\mathcal{M}$ to $\underline{\mathrm{Isom}}(\mathcal{O}_X,\mathcal{M})$. The action

$$\mathcal{O}_X^\times\times\underline{\mathrm{Isom}}(\mathcal{O}_X,\mathcal{M})\longrightarrow\underline{\mathrm{Isom}}(\mathcal{O}_X,\mathcal{M})$$

is given by

$$(g,f) \longmapsto f \circ g.$$

One can show $\underline{\mathrm{Isom}}(\mathcal{O}_X, \mathcal{M}) \times^{\mathcal{O}_X^\times} \mathcal{O}_X \cong \mathcal{M}$ and $\underline{\mathrm{Isom}}(\mathcal{O}_X, P \times^{\mathcal{O}_X^\times} \mathcal{O}_X) \cong P$ choose a trivializing cover for each. From this you get local isomorphisms to the trivial bundle or trivial torsor and show that they glue nicely on intersections.

From this we get:

Theorem: There is a bijection

$$H^1(X, \mathcal{O}_X^\times) \longleftrightarrow \left\{ \begin{matrix} \text{isomorphism classes} \\ \text{of line bundles on } X \end{matrix} \right\}. \qquad \square$$

The short exact sequence (exponential sheaf sequence)

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \xrightarrow{e^{2\pi i}} \mathcal{O}_X^\times \longrightarrow 1$$

gives rise to the exact sequence

$$\cdots \longrightarrow H^1(X, \mathbb{Z}_X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow \cdots.$$

This short exact sequence exists since the exponential defines a homomorphism $\mathcal{O}_X \longrightarrow \mathcal{O}_X^\times$. For surjectivity let $s \in \mathcal{O}_X^\times(U)$ take a cover of U by disks

$\mathcal{U}_i$. Then for each $\mathcal{U}_i$ we choose a logarithm and define $t_i := \log_{\mathcal{U}_i} s$. So we have that $s_{|\mathcal{U}_i} = e^{2\pi\sqrt{-1}\,t_i}$ and thus the exponential map is surjective. Some normalization of $\log_{\mathcal{U}_i}$ might be necessary.
We want to study the first group in the long exact sequence. Consider the exact sequence

$$\begin{tikzcd} & & & S^1 \arrow[d, equal] & \\ 0 \arrow[r] & \mathbb{Z}_X \arrow[r] & \mathbb{R}_X \arrow[r] & \mathcal{U}(1)_X \arrow[r] & 0 \end{tikzcd}$$

from which we get

$$\cdots \to H^0(X, \mathbb{R}_X) \to H^0(X, \mathcal{U}(1)_X) \to H^1(X, \mathbb{Z}_X) \to H^1(X, \mathbb{R}_X) \to \cdots.$$

X is connected and therefore we have a identification $H^0(X, \mathbb{R}_X) \cong \mathbb{R} \twoheadrightarrow \mathcal{U}(1) \cong H^0(X, \mathcal{U}(1)_X)$.
Hence $H^1(X, \mathbb{Z}_X) \to H^1(X, \mathbb{R}_X)$ is injective. Now we have

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R}_X \arrow[r] & C^\infty_{\mathbb{R}} \arrow[r, "d"] \arrow[dr] & \mathcal{E}^1_{\mathbb{R},cl} \arrow[r] \arrow[d] & 0 & \\ & & & A^1_{\mathbb{R}} \arrow[r] & A^2_{\mathbb{R}} \arrow[r] & 0 \end{tikzcd}$$

where $C^\infty_{\mathbb{R}}$ is the sheaf of real valued smooth functions and $\mathcal{E}^1_{\mathbb{R},cl}$ is the sheaf of real valued closed forms. The map d is associated with the differential of the de Rham complex of a real manifold. To see that d is surjective let $s \in \mathcal{E}^1_{\mathbb{R},cl}(\mathcal{U})$. Take $\mathcal{U}_i$ disks. By Poincaré lemma $s_{|\mathcal{U}_i}$ is exact. We define t_i by $s_{|\mathcal{U}_i} = dt_i$ and we have surjectivity. From this we have

$$\cdots \to C^\infty_{\mathbb{R}}(X) \to \mathcal{E}^1_{\mathbb{R},cl}(X) \to H^1(X, \mathbb{R}_X) \to H^1(X, C^\infty_{\mathbb{R}}) \to \cdots.$$

An element of $H^1(X, C^\infty_{\mathbb{R}})$ is represented by a cocycle

$f_{ij} \in C^\infty_{\mathbb{R}}(U_{ij})$. Choose a smooth partition of unity ψ_i subordinate to the open cover U_i. Let

$$\phi_i = \sum_k \psi_k f_{ik}.$$

We compute

$$\phi_i - \phi_j = \sum_k \psi_k (f_{ik} - f_{jk}) = *.$$

By the cocycle condition

$$* = \sum_k \psi_k f_{ij} = f_{ij} \sum_k \psi_k = *.$$

Since ψ_k is a partition of unity

$$* = f_{ij}.$$

So f_{ij} is a coboundary and $H^1(X, C^\infty_{\mathbb{R}}) = 0$.

From this we obtain a version of de Rham's theorem:

Theorem:

$$H^1(X, \mathbb{R}_X) \cong \{\text{closed real 1-forms}\}/\{\text{exact real 1-forms}\}.$$

By de Rham's theorem, $H^1(X, \mathbb{R}_X)$ is a $2g$-dimensional real vector space.

Proof: We have done most of the work, we just note that the image of $C^\infty_{\mathbb{R}}(X)$ in $\mathcal{E}^1_{\mathbb{R},ce}(X)$ is by definition the exact forms.

de Rham's theorem and what we have proven give us

$$H^1(X,\mathbb{R}_X) \cong H^1_{dR}(X) \cong H^1(X,\mathbb{R}).$$

For the dimension, consider the short exact sequence

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{O}_X^{hol} \longrightarrow \Omega_X^{1,hol} \longrightarrow 0.$$

We get a triangle

$$R\Gamma(X,\mathbb{C}_X) \longrightarrow R\Gamma(X,\mathcal{O}_X^{hol}) \longrightarrow R\Gamma(X,\Omega_X^{1,hol}) \longrightarrow \cdots.$$

Therefore

$$\chi(\mathbb{C}_X) - \chi(\Omega_X^{1,hol}) = \chi(\mathcal{O}_X^{hol})$$

$$2 - \dim_{\mathbb{C}} H^1(X,\mathbb{C}_X) + g - 1 = 1 - g$$

$$\dim_{\mathbb{R}} H^1(X,\mathbb{R}_X) = \dim_{\mathbb{C}} H^1(X,\mathbb{C}_X) = 2g. \qquad \square$$

Consider next $H^1(X,\mathbb{Z}_X) \subset H^1(X,\mathbb{R}_X)$. Like for $\mathbb{R}$ we have an isomorphism

$$H^1(X,\mathbb{Z}_X) \cong H^1(X,\mathbb{Z}).$$

$H^1(X,\mathbb{Z}_X)$ can be identified with the subgroup represented by closed 1-forms α such that

$$\int_\gamma \alpha \in \mathbb{Z}$$

for all closed loops $\gamma \subset X$. It is sufficient to check for $\gamma_1, \dots, \gamma_{2g}$ a homology basis of $H_1(X,\mathbb{Z})$. Therefore $H^1(X,\mathbb{Z}_X)$ forms a lattice inside $H^1(X,\mathbb{R}_X)$.

Theorem: $H^1(X,\mathbb{Z})$ forms a lattice inside $H^1(X,\mathcal{O}_X)$. Therefore the quotient is a g-dimensional complex torus.

Proof: Since $H^1(X,\mathbb{Z})$ is a lattice in $H^1(X,\mathbb{R})$, we need to construct an isomorphism $\pi: H^1(X,\mathbb{R}) \xrightarrow{\cong} H^1(X,\mathcal{O}_X)$ compatible with the maps $H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathbb{R})$. The map π is the one induced by $\mathbb{R}_X \longrightarrow \mathcal{O}_X$. It is not obvious, but we have the following isomorphisms [3]

$$H^1(X,\mathbb{R}) \cong \{\text{harmonic real 1-forms}\},$$
$$H^1(X,\mathcal{O}_X) \cong \{\text{antiholomorphic 1-forms}\}.$$

Given α we have $\pi(a) = \alpha^{(0,1)} \in H^1(X,\mathcal{O}_X)$.

$$\begin{tikzcd}
H^1(X,\mathbb{R}) \arrow[r, "\pi"] \arrow[d, "\cong"'] & H^1(X,\mathcal{O}_X) \arrow[d, "\cong"] \\
\{\text{h.r. 1-forms}\} \arrow[r, "\pi"] & \{\text{a.h. 1-forms}\}
\end{tikzcd}$$

$$f_1(z)dz + f_2(z)d\bar{z} \longmapsto f_2(z)d\bar{z}$$

The inverse is the map sending $\beta \in H^1(X,\mathcal{O}_X)$ to $\beta + \bar{\beta} \in H^1(X,\mathbb{R})$. $\square$

Now that we have shown that the image of $H^1(X,\mathcal{O}_X)$ in $H^1(X,\mathcal{O}_X^\times)$ is a torus. Recall

$$\begin{tikzcd}
H^1(X,\mathbb{Z}) \arrow[r, hook] \arrow[dr, hook] & H^1(X,\mathcal{O}_X) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^1(X,\mathcal{O}_X^\times) \arrow[r] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0. \\
& H^1(X,\mathbb{R}) & & \mathbb{Z} &
\end{tikzcd}$$

We can proceed to show that $Cl^0(X)$ coincides with the image of $H^1(X,\mathcal{O}_X)$. For this we start by defining the first Chern class.

Proposition: $\mathcal{M}$ lies in the image of $H^1(X,\mathcal{O}_X)$ iff $c_1(\mathcal{M})=0$. Where c_1 is the connecting homomorphism $H^1(X,\mathcal{O}_X^\times)\longrightarrow H^2(X,\mathbb{Z})$ and $c_1(\mathcal{M})=c_1([f_{ij}])$.

To understand c_1 better we will show another interpretation of it. We can map $c_1(\mathcal{M})$ to $H^2(X,\mathbb{C}_X)\cong H^2(X,\mathbb{C})\cong H^2_{dR,\mathbb{C}}(X)$ under the embedding $\mathbb{Z}_X\to\mathbb{C}_X$. Then $c_1(\mathcal{M})$ is a differential form. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_X \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_X^\times \arrow[r] \arrow[d, "\frac{1}{2\pi\sqrt{-1}} d\log"] & 1 \\
0 \arrow[r] & \mathbb{C}_X \arrow[r] & \mathcal{O}_X \arrow[r, "d"] & \Omega^1_X \arrow[r] & 0
\end{tikzcd}$$

where $d\log f = df/f$. We have

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[r] \arrow[d] & H^1(X,\mathcal{O}_X^\times) \arrow[r] \arrow[d] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d] & H^2(X,\mathcal{O}_X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[r] & H^1(X,\Omega^1_X) \arrow[r, "\cong"] & H^2(X,\mathbb{C}) \arrow[r] & \underbrace{H^2(X,\mathcal{O}_X)}_{=0} \arrow[r] & \cdots
\end{tikzcd}$$

from which we get that $c_1([f_{ij}])$ is the coboundary of

$$\omega_{ij} := \frac{1}{2\pi\sqrt{-1}} d\log(f_{ij}).$$

ω_{ij} is a holomorphic 1-form (can be thought of as smooth). By a partition of unity argument we can find α_i such that $\omega_{ij}=\alpha_i-\alpha_j$. Let $\beta_i=d\alpha_i$. When restricting to U_{ij} we get

$$\beta_i-\beta_j = d\alpha_i - d\alpha_j = d\omega_{ij} =$$

$$= \frac{1}{2\pi\sqrt{-1}} dd\log(f_{ij}) = 0.$$

So the β_i glue nicely. β is a 2-form and X is 2-dimensional (real) so we can integrate β over X to get a number in $\mathbb{C}$ because X is compact. Since there is an isomorphism $H^2(X,\mathbb{C}) \cong \mathbb{C}$ we can treat $c_1(M)$ as a number.

Proposition: $c_1(\mathcal{O}(D)) = \pm \deg D$ for a divisor D.

Proof: Suppose $X = \mathbb{P}^1_{\mathbb{C}}$, $D = 1 \cdot 0$. Then

$$\begin{array}{ccccccccc} \cdots \to & H^1(X,\mathcal{O}_X) & \longrightarrow & H^1(X,\mathcal{O}_X^\times) & \longrightarrow & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathcal{O}_X) & \to \cdots . \\ & \| & & \wr\| & & \wr\| & & \| & \\ & 0 & & \mathbb{Z}\cdot\mathcal{O}(1) & & \mathbb{Z} & & 0 & \end{array}$$

So $c_1(\mathcal{O}(1)) = \pm 1 = \deg \mathcal{O}(1)$. We claim this case is sufficient.

Let X be arbitrary again. Suppose $D = n \cdot x$. By Riemann-Roch if $n >> 0$ is large enough there exists $f : X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ such that $\mathcal{O}(D) = f^*\mathcal{O}(1)$ and $D = f^{-1}(0)$ as sets. Therefore

$$\begin{array}{rcl} c_1(\mathcal{O}(D)) = f^* c_1(\mathcal{O}(1)) \in H^2(X,\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \\ f^* \uparrow \quad (*) & & \uparrow \deg f = n \\ H^2(\mathbb{P}^1_{\mathbb{C}},\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} . \end{array}$$

Provided that $(*)$ commutes, we are done. But we can view $\mathcal{O}(1)$ as the line bundle associated to x as a divisor and then $f^*\mathcal{O}(x)$ is the line bundle associated to $f^*(x)$. So then we have $\deg f^*(x) = \deg f \cdot \deg(x)$ which implies $\deg f = n$. $\square$

This completes the proof of the first theorem.

We want to give another description of $J(X)$. We have

$$H^1(X,\mathcal{O}_X) \xrightarrow{\phi} H^0(X,\Omega^1_X)^*$$

$$\omega \longmapsto \left(\beta \mapsto \int_X \beta\wedge\omega\right).$$

where ω is viewed as an antiholomorphic 1-form. We claim this is an isomorphism. Observe that both sides have dimension g and since if $\beta\neq 0$ then

$$\int_X \beta\wedge\bar{\beta}\neq 0$$

and we have that the kernel is trivial. Under ϕ the image of $H^1(X,\mathbb{Z})$ in $H^1(X,\mathcal{O}_X)$ is mapped to the image of $H_1(X,\mathbb{Z})$ in $H^0(X,\Omega^1_X)^*$ by

$$\gamma \longmapsto \int_\gamma - .$$

Therefore

$$J(X)\cong H^0(X,\Omega^1_X)^*/H_1(X,\mathbb{Z}).$$

We carry on to the final part where we discuss the Abel-Jacobi map.

Definition: Fix a basepoint $p_0 \in X$. Define the Abel-Jacobi map

$$\alpha : X \to J(X)$$
$$x \mapsto x - p_0.$$

Proposition: α is holomorphic.

Definition: The n-th symmetric product of X is

$$S^n X = X \times \cdots \times X / S_n.$$

If $z_1, \dots, z_n$ are local coordinates of X^n then the elemetary symmetric functions give us local coordinates $\sigma_i(z_1, \dots, z_n)$ on $S^n X$. Thus $S^n X$ is a complex manifold. $S^n X$ corresponds to effective divisors of degree n since $(x_1, \dots, x_2) \mapsto x_1 + \dots + x_n$ is stable under permutation and $x_1 + \dots + x_n$ is effective. Define

$$\alpha_n : S^n X \longrightarrow J(X)$$
$$x_1 + \dots + x_n \longmapsto x_1 - p_0 + \cdots + x_n - p_0.$$

Theorem: (i) (Abel) If $D \in S^n X$ then the fiber $\alpha_n^{-1}\alpha_n(D)$ consists of all effective divisors linearly equivalent to D, i.e. D' such that $D-D'$ is principal.
(ii) (Jacobi) α_g is surjective.

Corollary: If X is an elliptic curve, i.e. genus 1, then $X \cong J(X)$.

Proof: Note that $\alpha_1 = \alpha$. So α is holomorphic and surjective. Let $p \in X$. Then $H^0(X, \mathcal{O}(p))$ is a vector space over $\mathbb{C}$ and by Riemann-Roch is of dimension 1 and isomorphic to $\mathbb{C}$. So p is the only effective divisor linearly equivalent to p. Hence α has degree 1 and is then an isomorphism. $\square$

Corollary: $J(X)$ and $S^g X$ are bimeromorphically equivalent, i.e. there exists a meromorphic map from one the other which admits a meromorphic inverse.

References

[1] Arapura - The Jacobian of a Riemann Surface

[2] Griffiths & Harris - Principles of Algebraic Geometry

[3] Arapura - Riemann's Inequality and Riemann-Roch